



IT Basics 6

An Introduction to Computers' Logic

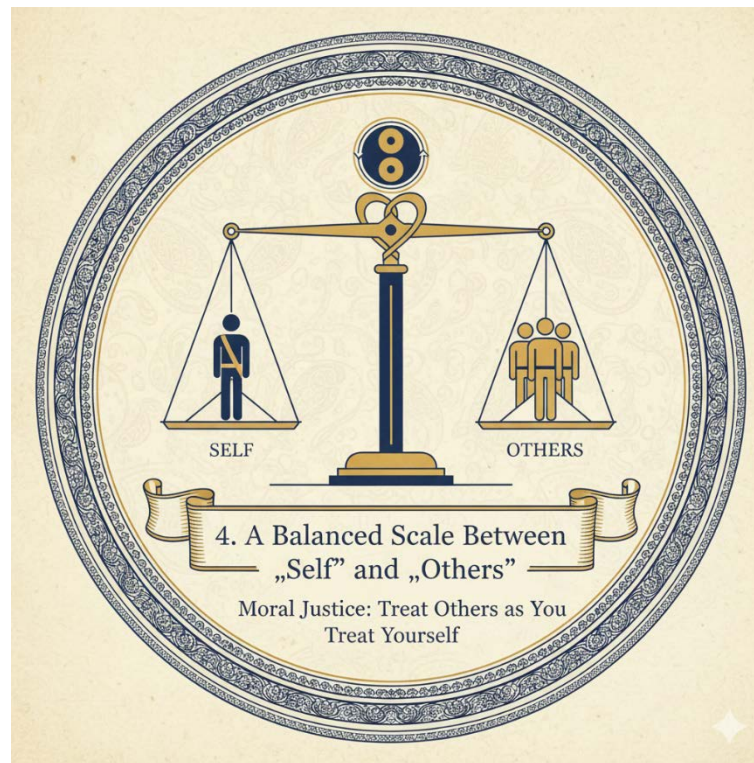
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Instead of “introduction”

“Treat others the way you want to be treated” !



Computers' Logic Basics – An Introduction

VLSI (Very Large Scale Integration)

- It represents the manufacturing process of an IC (Integrated Circuit) by combining thousands and hundreds of thousands of logical gates (transistors) on a single silicon chip.
- The microprocessor is a VLSI equipment
- The beginnings of VLSI: '70



Computers' Logic Basics – An Introduction

ULSI (Ultra Large Scale Integration) – chips with more than 1 million components

For example, one **Intel® Xeon Core i7 E3 Broadwell** has 3.2 billion transistors on a die of only 133 mm². A **Core i9** (14th generation) has between 20-25 billion transistors per chip.

Intel's supercomputing GPU, Ponte Vecchio, and Nvidia's GPUs are among the leaders. As of Nvidia's latest GPU, based on the Blackwell architecture, has set new records in transistor count, boasting **208 billion** transistors per unit.



Computers' Logic Basics – An Introduction

In a A17 Pro chipset from an iPhone there are **19 billion transistors** and in an Apple M1 Ultra there are integrated more than **114 billion transistors**.



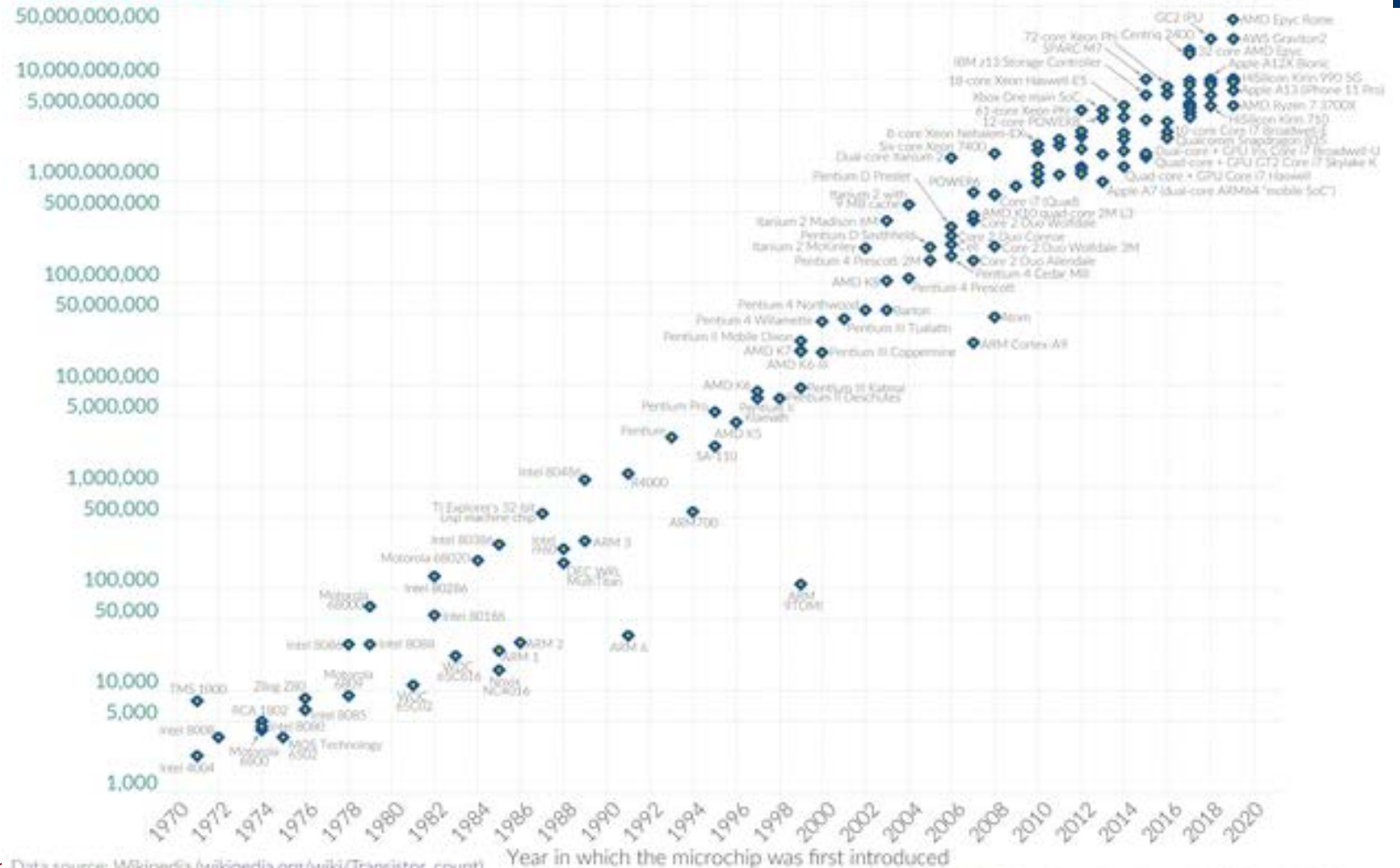
Gordon Moore' Law

Moore's Law: The number of transistors on microchips doubles every two years

Our World
in Data

Moore's law describes the empirical regularity that the number of transistors on integrated circuits doubles approximately every two years. This advancement is important for other aspects of technological progress in computing – such as processing speed or the price of computers.

Transistor count



The exponential growth in computing power!

1 The accelerating pace of change ...



2 ... and exponential growth in computing power ...

Computer technology, shown here climbing dramatically by powers of 10, is now progressing more each hour than it did in its entire first 90 years

COMPUTER RANKINGS

By calculations per second per \$1,000



Analytical engine
Never fully built, Charles Babbage's invention was designed to solve computational and logical problems



Colossus
The electronic computer, with 1,500 vacuum tubes, helped the British crack German codes during WW II



UNIVAC I
The first commercially marketed computer, used to tabulate the U.S. Census, occupied 943 cu. ft.

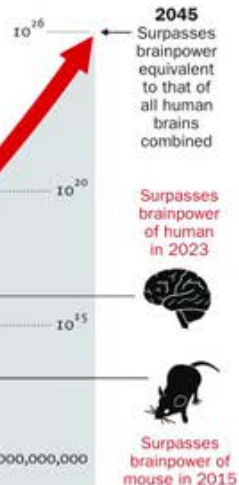


Apple II
At a price of \$1,298, the compact machine was one of the first massively popular personal computers



Power Mac G4
The first personal computer to deliver more than 1 billion floating-point operations per second

3 ... will lead to the Singularity



Source: <http://content.time.com/time/interactive/0,31813,2048601,00.html>

Introduction

- Digital components
- Digital electronics (+5V, -5V), (0V, -5V), (0V, +5V)

0 and 1 can be represented by +5V and -5V, or 0V and -5V, or 0V and +5V

Introduction

Boole's algebra

Basic operations:

- Disjunction (OR)
- Conjunction (AND)
- Negation (NOT)

Truth tables – disjunction, conjunction, negation

p	q	p AND q
T	T	T
T	F	F
F	T	F
F	F	F

p	q	p OR q
T	T	T
T	F	T
F	T	T
F	F	F

p	q	p AND q
1	1	1
1	0	0
0	1	0
0	0	0

p	~ p
1	0
0	1

Boole's algebra – fundamental theorems

1. Identities and null elements:

- There is element 0 (zero) as **prime element** with the properties: $x \cap 0 = 0$ and $x \cup 0 = x$
- There is element 1 (one) denoted **last element** with the properties: $x \cap 1 = x$ and $x \cup 1 = 1$

2. Uniqueness:

- Element 1 is unique
- Element 0 is unique

3. Complements - for every x in K there is a unique element $\overline{x}$ such that:

$$x \cap \overline{x} = 0$$

$$x \cup \overline{x} = 1$$

Boole's algebra – fundamental theorems (cont.)

4. Double negation theorem: $\overline{\overline{x}} = x$

5. Absorption theorems:

- $x \cup (x \cap y) = x$
- $x \cap (x \cup y) = x$

6. De Morgan theorems:

$$\overline{x \cup y} = \overline{x} \cap \overline{y}$$

$$\overline{x \cap y} = \overline{x} \cup \overline{y}$$

Boole's algebra – fundamental theorems (cont.)

7. Idempotency:

$$x \cup x \cup \dots \cup x = x$$

$$x \cap x \cap \dots \cap x = x$$

8. Commutativity, associativity and distributivity:

- $x \cup y = y \cup x$
- $x \cup (y \cup z) = (x \cup y) \cup z$
- $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
- $x \cap y = y \cap x$
- $x \cap (y \cap z) = (x \cap y) \cap z$
- $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

Boolean functions: existence and uniqueness

$$B_2 = \{0, 1\}$$

$f : B_2 \rightarrow B_2$ single variable function

$f : B_2 \times B_2 \rightarrow B_2$ double variable function

$f : \underbrace{B_2 \times B_2 \times \cdots \times B_2}_{n \text{ ori}} \rightarrow B_2$ n variable function

Primary definitions

We call an *elementary product/elementary sum* a *product/sum* of variables *and/or* their negations

We call *disjunctive canonical form (DCF)* of a logical relation, an equivalent relation (with the same truth value) which is a *sum of elementary products* constructed with the same variables as the initial relation, each product containing all variables (in normal or complementary form)

Primary definitions

We call *conjunctive canonical form (CCF)* of a logical function, an equivalent relation (with the same truth value) which is a *product of elementary sums* constructed with the same variables as the initial relation, each sum containing all the possible variables (in normal or complementary form)

DCF for one variable function

Let be $f : B_2 \rightarrow B_2$ a boolean function with one variable and a, b two boolean constants

$f(x) = ax \cup b\bar{x}$ - disjunctive canonical form

$f(x) = (a \cup x)(b \cup \bar{x})$ - conjunctive canonical form

These functions are uniquely determined

Replacing $x = 1, x = 0$ in the relation of $f(x)$:

$$\begin{cases} f(1) = a \bullet 1 \cup b \bullet \bar{1} = a \bullet 1 \cup b \bullet 0 = a \cup 0 = a \\ f(0) = a \bullet 0 \cup b \bullet \bar{0} = a \bullet 0 \cup b \bullet 1 = 0 \cup b = b \end{cases}$$

$$DCF : f(x) = f(1) \bullet x \cup f(0) \bullet \bar{x}$$

CCF for one variable function

$f(x) = (a \cup x)(b \cup \bar{x})$ - conjunctive canonical form

Replacing $x = 1, x = 0$ in the relation of $f(x)$:

$$\begin{cases} f(1) = (a \cup 1) \bullet (b \cup \bar{1}) = (a \cup 1) \bullet (b \cup 0) = 1 \bullet b = b \\ f(0) = (a \cup 0) \bullet (b \cup \bar{0}) = (a \cup 0) \bullet (b \cup 1) = a \bullet 1 = a \end{cases}$$

$$CCF : f(x) = (f(0) \cup x) \bullet (f(1) \cup \bar{x})$$

Proof of existence (DCF)

Considering $f(x) = f(1) \bullet x \cup f(0) \bullet \bar{x}$ in DCF and replacing x with 0 and 1.

$$\begin{cases} f(1) = f(1) \bullet 1 \cup f(0) \bullet \bar{1} = f(1) \bullet 1 \cup f(0) \bullet 0 = f(1) \\ f(0) = f(1) \bullet 0 \cup f(0) \bullet \bar{0} = f(1) \bullet 0 \cup f(0) \bullet 1 = f(0) \end{cases}$$

Proof of existence (CCF)

Considering $f(x) = (f(0) \cup x) \bullet (f(1) \cup \bar{x})$ in CCF and replacing x with values 0 and 1:

$$\begin{cases} f(1) = (f(0) \cup 1) \bullet (f(1) \cup \bar{1}) = (f(0) \cup 1)(f(1) \cup 0) = 1 \bullet f(1) = f(1) \\ f(0) = (f(0) \cup 0) \bullet (f(1) \cup \bar{0}) = (f(0) \cup 0)(f(1) \cup 1) = f(0) \bullet 1 = f(0) \end{cases}$$

DCF for two variables function

Let be $f : B_2 \times B_2 \rightarrow B_2$ A boolean function with two variables and a,b,c,d boolean constants

$$f(x, y) = axy \cup bx\bar{y} \cup c\bar{x}y \cup d\bar{x}\bar{y} - \text{DCF}$$

$$f(x, y) = (a \cup x \cup y)(b \cup x \cup \bar{y})(c \cup \bar{x} \cup y)(d \cup \bar{x} \cup \bar{y}) - \text{CCF}$$

Considering the DCF and replacing $x = 1, x = 0$ in the relation of $f(x)$. We will have :

$$\begin{cases} a = f(1,1) \\ b = f(1,0) \\ c = f(0,1) \\ d = f(0,0) \end{cases}$$

$$\text{DCF} : f(x, y) = f(1,1)xy \cup f(1,0)x\bar{y} \cup f(0,1)\bar{x}y \cup f(0,0)\bar{x}\bar{y}$$

CCF for a function with two variables

$$f(x, y) = (a \cup x \cup y)(b \cup x \cup \bar{y})(c \cup \bar{x} \cup y)(d \cup \bar{x} \cup \bar{y}) - \text{CCF}$$

Replacing in this equation with $x = 1, x = 0$ we have:

$$\begin{cases} a = f(0,0) \\ b = f(0,1) \\ c = f(1,0) \\ d = f(1,1) \end{cases}$$

$$\text{CCF} : f(x, y) = (f(0,0) \cup x \cup y)(f(0,1) \cup x \cup \bar{y})(f(1,0) \cup \bar{x} \cup y)(f(1,1) \cup \bar{x} \cup \bar{y})$$

Proof of existence for DCF

$$f(x, y) = f(1,1)xy \cup f(1,0)x\bar{y} \cup f(0,1)\bar{x}y \cup f(0,0)\bar{x}\bar{y} - \text{DCF}$$

Considering the equation and replacing $x = 1, x = 0, y = 1, y = 0$ we get :

$$\left\{ \begin{array}{l} x = y = 1 \Rightarrow f(1,1) = f(1,1) \bullet 1 \bullet 1 \cup f(1,0) \bullet 1 \bullet \bar{1} \cup f(0,1) \bullet \bar{1} \bullet 1 \cup f(0,0) \bullet \bar{1} \bullet \bar{1} = f(1,1) \\ x = 1, y = 0 \Rightarrow f(1,0) = f(1,1) \bullet 1 \bullet 0 \cup f(1,0) \bullet 1 \bullet \bar{0} \cup f(0,1) \bullet \bar{1} \bullet 0 \cup f(0,0) \bullet \bar{1} \bullet \bar{0} = f(1,0) \\ x = 0, y = 1 \Rightarrow f(0,1) = f(1,1) \bullet 0 \bullet 1 \cup f(1,0) \bullet 0 \bullet \bar{1} \cup f(0,1) \bullet \bar{0} \bullet 1 \cup f(0,0) \bullet \bar{0} \bullet \bar{1} = f(0,1) \\ x = y = 0 \Rightarrow f(0,0) = f(1,1) \bullet 0 \bullet 0 \cup f(1,0) \bullet 0 \bullet \bar{0} \cup f(0,1) \bullet \bar{0} \bullet 0 \cup f(0,0) \bullet \bar{0} \bullet \bar{0} = f(0,0) \end{array} \right.$$

Proof of existence for CCF

$$CCF : f(x, y) = (f(0,0) \cup x \cup y)(f(0,1) \cup x \cup \bar{y})(f(1,0) \cup \bar{x} \cup y)(f(1,1) \cup \bar{x} \cup \bar{y})$$

Considering the above equation and replacing $x=1, x=0, y=1, y=0$ we get :

$$\left\{ \begin{array}{l} x = y = 1 \Rightarrow f(1,1) = (f(0,0) \cup 1 \cup 1)(f(0,1) \cup 1 \cup \bar{1})(f(1,0) \cup \bar{1} \cup 1)(f(1,1) \cup \bar{1} \cup \bar{1}) = f(1,1) \\ x = 1, y = 0 \Rightarrow f(1,0) = (f(0,0) \cup 1 \cup 0)(f(0,1) \cup 1 \cup \bar{0})(f(1,0) \cup \bar{1} \cup 0)(f(1,1) \cup \bar{1} \cup \bar{0}) = f(1,0) \\ x = 0, y = 1 \Rightarrow f(0,1) = (f(0,0) \cup 0 \cup 1)(f(0,1) \cup 0 \cup \bar{1})(f(1,0) \cup \bar{0} \cup 1)(f(1,1) \cup \bar{0} \cup \bar{1}) = f(0,1) \\ x = y = 0 \Rightarrow f(0,0) = (f(0,0) \cup 0 \cup 0)(f(0,1) \cup 0 \cup \bar{0})(f(1,0) \cup \bar{0} \cup 0)(f(1,1) \cup \bar{0} \cup \bar{0}) = f(0,0) \end{array} \right.$$

Truth tables

We can associate any logic function with a truth table

$$f(x, y) = \overline{\overline{x}y} \cup xy$$

x	y	$\overline{\overline{x} \bullet \overline{y}}$	$x \bullet y$	$f(x, y)$	$\overline{f(x, y)}$
0	0	1	0	1	0
0	1	0	0	0	1
1	0	0	0	0	1
1	1	0	1	1	0

Truth tables (cont.)

Reciprocally, if we have a truth table we may determine the logic function

From the truth table $f = 1$ for the following values for x and y :

$$\begin{cases} x = y = 0 \\ x = y = 1 \end{cases}$$

or

$$\begin{cases} \bar{x} = 1, \bar{y} = 1 \\ x = y = 1 \end{cases}$$

$$f_1(x, y) = \bar{x} \bullet \bar{y} \cup x \bullet y$$

Truth tables (cont.)

Considering the values of x and y for $f = 0$:

$$\begin{cases} x = 0, y = 1 \\ x = 1, y = 0 \end{cases}$$

or

$$\begin{cases} x = 0, \bar{y} = 0 \\ \bar{x} = 0, y = 0 \end{cases}$$

$$f_2(x, y) = (x \cup \bar{y})(\bar{x} \cup y)$$

Truth tables (cont.)

The functions must be equal :

$$\begin{aligned} f_2(x, y) &= (x \cup \bar{y})(\bar{x} \cup y) = x \bullet \bar{x} \cup x \bullet y \cup \bar{y} \bullet \bar{x} \cup \bar{y} \bullet y = \\ &= x \bullet y \cup \bar{x} \bullet \bar{y} = f_1(x, y) \end{aligned}$$

Logical functions representation

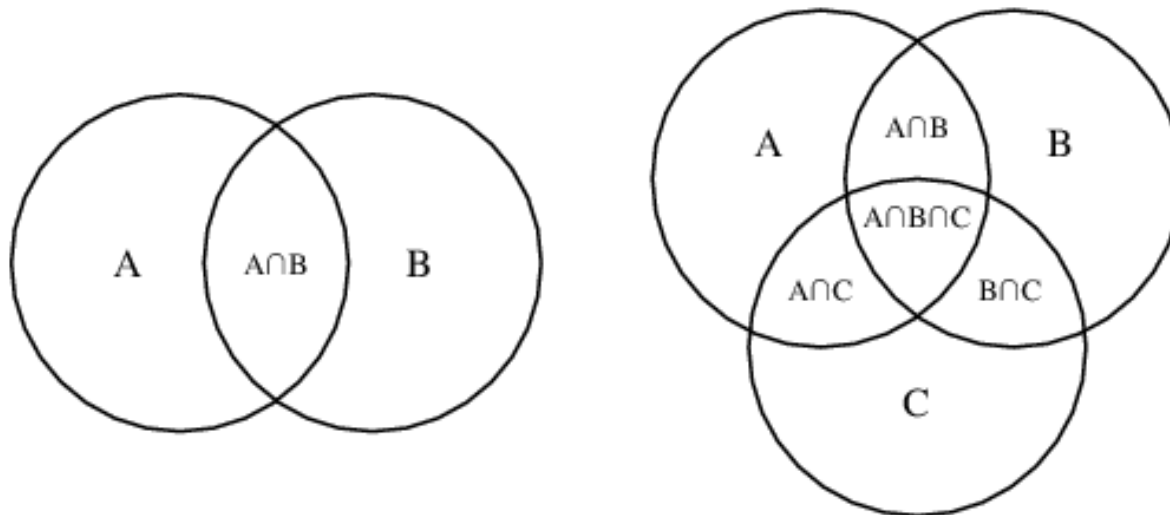
Canonical forms:

- ♦ ***Minterm*** form (DCF –disjunctive canonical form) – SUM of products – variables or their complements in a ***minterm*** are connected by boolean operation **AND**, and the ***minterms*** are connected by **OR**.
- ♦ ***Maxterm*** form (CCF –conjunctive canonical form) – PRODUCT of sums – variables or their complements in a ***maxterm*** are connected by **OR**, and ***maxterms*** are connected by **AND**.

Another form of representation of a boolean function is the graphical one, using Venn diagrams

Venn diagram example

- ◆ A Venn diagram is a schematic diagram used in logic theory to depict collections of sets and represent their relationships.



Minterms/maxterms for a function with two variables

Two variable function			
x	y	Minterms m_i	Maxterms M_i
0	0	$m_0 = \overline{x}\overline{y}$	$M_0 = x \cup y$
0	1	$m_1 = \overline{x}y$	$M_1 = x \cup \overline{y}$
1	0	$m_2 = x\overline{y}$	$M_2 = \overline{x} \cup y$
1	1	$m_3 = xy$	$M_3 = \overline{x} \cup \overline{y}$

Minterms/maxterms for a function with three variables

Three variable function				
x	y	z	Minterms m_i	Maxterms M_i
0	0	0	$m_0 = \overline{x}\overline{y}\overline{z}$	$M_0 = x \cup y \cup z$
0	0	1	$m_1 = \overline{x}\overline{y}z$	$M_1 = x \cup y \cup \overline{z}$
0	1	0	$m_2 = \overline{x}y\overline{z}$	$M_2 = x \cup \overline{y} \cup z$
0	1	1	$m_3 = \overline{x}yz$	$M_3 = x \cup \overline{y} \cup \overline{z}$
1	0	0	$m_4 = x\overline{y}\overline{z}$	$M_4 = \overline{x} \cup y \cup z$
1	0	1	$m_5 = x\overline{y}z$	$M_5 = \overline{x} \cup y \cup \overline{z}$
1	1	0	$m_6 = xy\overline{z}$	$M_6 = \overline{x} \cup \overline{y} \cup z$
1	1	1	$m_7 = xyz$	$M_7 = \overline{x} \cup \overline{y} \cup \overline{z}$

Minterms/maxterms - properties

Minterms are formed by the combination of the variables (or its' complements) for which the function has the value of 1.

Maxterms are formed by the combination of the variables (or its' complements) for which the function has the value of 0.

$$m_i = \overline{M}_i$$

$$M_i = \overline{m}_i, \forall i$$

Minterms/maxterms - properties (cont.)

P1. The logic product between two terms m_i and m_j ($i \neq j$) of a logic function of n variables has the value of zero:

$$m_i \cdot m_j = 0, \forall i \neq j$$

P2. The logic sum between two terms M_i and M_j ($i \neq j$) of a logic function of n variables has the value of 1:

$$M_i \cup M_j = 1, \forall i \neq j$$

Minterms/maxterms - properties (cont.)

P3. A logic function of n variables may be represented by a logic sum of minterms $\mathbf{m}_i$ (respectively, by a logic product of maxterms $\mathbf{M}_i$):

$$f(x_1, \dots, x_n) = \bigcup_{i=0}^{2^n-1} (\alpha_i \cdot m_i)$$

$$f(x_1, \dots, x_n) = \bigcap_{i=0}^{2^n-1} (\alpha_i \cup M_i), \alpha_i \in \{0,1\}$$

Minterms/maxterms - properties (cont.)

P4. The complement of a logic function of n variables written in CCF may be expressed uniquely using the relation:

$$f(x_1, \dots, x_n) = \bigcup_{i=0}^{2^n-1} (\overline{\alpha_i} \cdot m_i)$$

and the complement of a logic function of n variables

written in DCF may be expressed uniquely using the relation :

$$f(x_1, \dots, x_n) = \bigcap_{i=0}^{2^n-1} (\overline{\alpha_i} \cup M_i), \alpha_i \in \{0,1\}$$

Minterms/maxterms - properties (cont.)

P5. If a logic function of n variables written in DCF is containing 2^n distinct terms of n variables then it is equal with 1.
In the same conditions, if the function is written in CCF, then it is equal with 0.

$$f(x_1, \dots, x_n) = \bigcup_{i=0}^{2^n-1} m_i = 1 \text{ (DCF)}$$

$$f(x_1, \dots, x_n) = \bigcap_{i=0}^{2^n-1} M_i = 0 \text{ (CCF)}$$

Minterms/maxterms - properties (cont.)

P6. Any minterm m_i of a logic function of n variables written in **DCF** is equal with the logic product of $2^n - 1$ M_j terms, respectively any maxterm M_i of a logic function of n variables written in **CCF** is equal with the logic sum of $2^n - 1$ m_j terms:

$$m_i = \bigcap_{j \neq i} M_j, j = 0 \dots 2^n - 1$$

$$M_i = \bigcup_{j \neq i} m_j, j = 0 \dots 2^n - 1$$

Logic functions of 2 variables

In the case of the 2 variable logic functions we may have the following canonical forms:

$$f(x, y) = \alpha_0 m_0 \cup \alpha_1 m_1 \cup \alpha_2 m_2 \cup \alpha_3 m_3 \text{ (DCF)}$$

$$f(x, y) = (\alpha_0 \cup M_0)(\alpha_1 \cup M_1)(\alpha_2 \cup M_2)(\alpha_3 \cup M_3) \text{ (CCF)}$$

From here, it results 16 two-variable functions, in the minterm/maxterm formats, considering the 16 possible combinations for $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$

2 variables logic functions

$$f_0 = 0 - \text{ZERO (FALSE)}$$

$$f_1 = xy - \text{AND}$$

$$f_2 = x\bar{y} - \text{INHIBITION} \mid x \text{ inhibits } y$$

$$f_3 = x - \text{DENTITY} \quad \text{or TRANSFER}$$

$$f_4 = \bar{x}y - \text{INHIBITION} \mid y \text{ inhibits } x$$

$$f_5 = y - \text{IDENTITY} \quad \text{or TRANSFER}$$

$$f_6 = \bar{x}y \cup x\bar{y} - \text{XOR} \quad \text{Exclusive OR} \mid 1 \text{ if } x \neq y$$

$$f_7 = x \cup y - \text{OR}$$

2 variables logic functions (cont.)

$$f_8 = \overline{x \cup y} = \overline{\overline{x} \overline{y}} - \text{NOR}$$

$$f_9 = \overline{\overline{x} \overline{y}} \cup xy - \text{COINCIDENCE} \quad | \text{ 1 if } x=y$$

$$f_{10} = \overline{y} - \text{NOT}$$

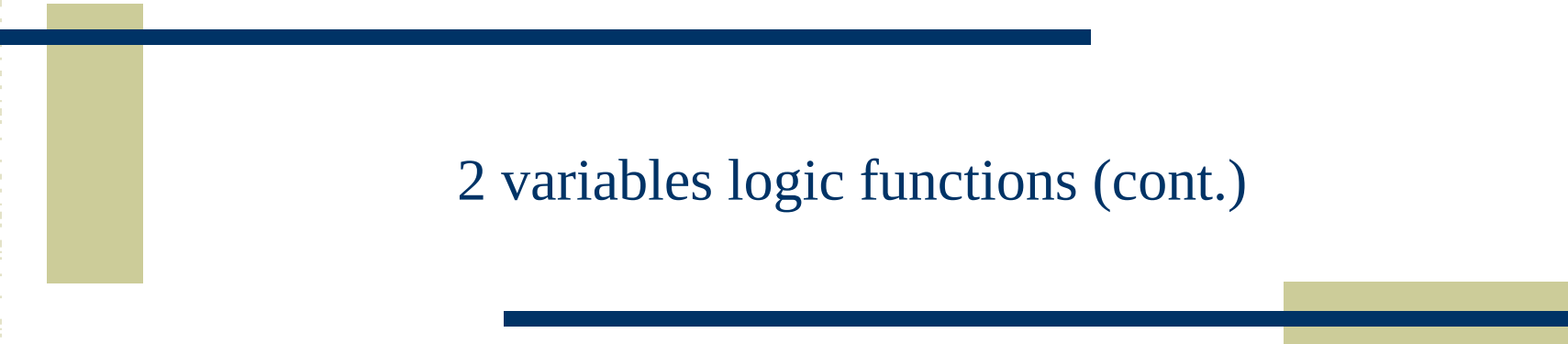
$$f_{11} = x \cup \overline{y} - \text{IMPLICATION}$$

$$f_{12} = \overline{x} - \text{NOT}$$

$$f_{13} = \overline{\overline{x} \cup y} - \text{IMPLICATION}$$

$$f_{14} = \overline{\overline{x} \cap \overline{y}} = \overline{\overline{x} \cup y} - \text{NAND}$$

$$f_{15} = 1 - \text{TRUE}$$



2 variables logic functions (cont.)

You may also check this website:

<https://www.allaboutcircuits.com/technical-articles/16-boolean-logic-functions-of-2-input-system/>

Logic function representation in DCF - example

Let be the function : $f(x, y, z) = \bar{x} \cup y\bar{z}$

Put it in the DCF form.

$$\begin{aligned} f(x, y, z) &= \bar{x}(y \cup \bar{y}) \cup y\bar{z}(x \cup \bar{x}) = \bar{x}y \cup \bar{x}\bar{y} \cup xy\bar{z} \cup \bar{x}y\bar{z} = \\ &\bar{x}y(z \cup \bar{z}) \cup \bar{x}\bar{y}(z \cup \bar{z}) \cup xy\bar{z} \cup \bar{x}y\bar{z} = \bar{x}yz \cup \bar{x}y\bar{z} \cup \bar{x}\bar{y}z \cup \bar{x}\bar{y}\bar{z} \cup xy\bar{z} \cup \bar{x}y\bar{z} = \\ &m_0 + m_1 + m_2 + m_3 + m_6 \end{aligned}$$